

Chapter 11

Advection-Diffusion Equations and Turbulence

11.1 Rationale

Why should we deal with advection-diffusion equations? They represent, after all, physical processes, which are the domain of the physical oceanographers. The problem is that a significant part of chemical oceanography involves the interpretation of the distributions of geochemicals in the water column, and these distributions are the result of the combined effects of physical, biological, chemical and geological processes. These processes are intimately intertwined in a way which requires us to “solve the physics” before we can begin to learn anything about the chemistry. In fact, there is an active branch of geochemistry which involves the use of geochemical tracer distributions to make inferences about ocean circulation and mixing. That is, some people make a living from this kind of stuff. The processes involved play a fundamental role in the ocean’s ability to exchange, sequester and transport heat, carbon dioxide and other biologically important properties. This in turn feeds back into the role of the oceans in regulating climate, global primary production and all sorts of socially relevant things.

Put another way, you can think of the water column distributions as being affected by physical and biogeochemical processes:

$$C(\vec{x}, t) = P(C) + J(\vec{x}, t) \quad (11.1)$$

where the space- and time-varying property of interest (C) is affected by some physical redistribution processes (the operator P) and some *biogeochemical* source/sink/transformation process (J). We somehow must deconvolve the physical processes to understand the biogeochemical processes. In this lecture, we will discuss the general character of open ocean transport processes and the concept of turbulent diffusion. The reason why we do this is because the nature of turbulent diffusion is not as clear cut and fundamental as is *molecular* diffusion. Yet it is an ugly necessity, for it involves a parameterization of processes that we have no hope of resolving directly, and which are

so ubiquitous and important, that we cannot ignore. Our hope is to at least make you aware of the philosophical underpinnings of the concept, the strengths and weaknesses of the approach, and the caveats that need be kept in mind when dealing with turbulent diffusivities. At least that way, you may have a better idea of why things appear the way they do.

11.2 The Basic Equation

Our starting point is really rather basic: we start with the Daltonian concept of the conservation of mass. Also, we'll only do this in one dimension. If you are really into pain, you can do this in three. We are assuming that the conservation of mass in one dimension can be written quite simply as:

$$\frac{dC}{dt} = \frac{\partial}{\partial x} \left(D \frac{\partial C}{\partial x} \right) + J \quad (11.2)$$

which can be stated in words as “the time rate of change of *stuff* in a fluid parcel is equal to the spatial rate of change of the (molecular) diffusive flux plus any *in situ* production”. The first term (the one on the left hand side) is often called the *Lagrangian derivative* or the “complete derivative”, since it quantifies the time rate of change following the fluid parcel. We *have* to express our conservation law in this form, since it is only from the perspective of the fluid parcel that we can guarantee the conservation of mass. The second term is simply the molecular diffusive flux divergence (D is the molecular diffusivity, which for most substances is of order 10^{-9} m²/s in water). That is, mass will only accumulate in a fluid parcel if the amount of material diffusing into it from one side exceeds or is less than the amount diffusing out. In order for this to occur, the diffusive flux must change with distance. With the assumption that the molecular diffusivity is constant in space (a reasonable assumption in most systems) the above equation can also be expressed in the possibly more familiar form as:

$$\frac{dC}{dt} = D \frac{\partial^2 C}{\partial x^2} + J \quad (11.3)$$

but we'll stick the other form for now. Finally, the last term J is there to represent possible source or sink processes, such as biological production/consumption or radioactive production/decay. Its form will remain unspecified here, but may be dependent on space, time, C and possibly other variables.

Now it is more common to view things in a space-fixed frame of reference, rather than a framework which follows fluid parcels. This space-fixed frame is referred to as an *Eulerian* (pronounced “oil-air-ian”) frame of reference, and we can translate to this frame of reference by taking partial derivatives:

$$\frac{dC}{dt} = \frac{\partial C}{\partial t} + \frac{\partial C}{\partial x} \frac{\partial x}{\partial t} = \frac{\partial C}{\partial t} + \frac{\partial}{\partial x}(uC) \quad (11.4)$$

which is simply to state that the rate of change in *stuff* in a fluid parcel is the sum of the time variation in the overall distribution (the first term) and the downstream change of *stuff* associated with its displacement. Another way of saying this is to recognize that the change in *stuff* at a location is the sum of the time rate of change of the distribution summed with the divergence of the *advective flux*. (The advective flux is just the concentration times the velocity). We can then rewrite our conservation equation as:

$$\frac{\partial C}{\partial t} = -\frac{\partial}{\partial x}(uC) + \frac{\partial}{\partial x}\left(D\frac{\partial C}{\partial x}\right) + J \quad (11.5)$$

The above equation is mathematically and physically correct in all respects, and if we can define the terms of this equation with adequate precision in both space and time, we could describe the evolution of C throughout the ocean. The trouble is, we can't.

11.3 Reynolds Decomposition

Here's where things get a little complicated. So far, what we have said is fundamentally sound and quantifiable. The problem is that we cannot know, measure or resolve all of the fluid motions that we *know* influence chemical properties in the ocean. Think of it this way: in order to completely describe the motion of the *stuff* that we have been talking about, we need to take into account movement on scales ranging from the molecular (order 10^{-8} m) to ocean basin (order 10^7 m), for all of these motions can contribute to changes in, say, oxygen concentrations. We are talking about 10^{15} orders of magnitude. This is clearly not possible. However, it is possible to describe the molecular diffusion of a dye in a glass of water (at least ideally) without knowing the individual motions of individual molecules: there are fundamental statistical mechanical (or thermodynamical) laws that can be derived to describe the behavior in a *statistical* sense. The hope is that we can do something along those lines.

We begin by thinking of the fluid motion as consisting of two parts: one which is the large scale, mean flow, and the other as being some small scale, randomly fluctuating component. This, we should warn you, is the fundamental, critical step in the development *to which you must pay careful attention*. It implies that there is some kind of **scale separation** that you can divide the motions into: large (mean) and small (random). It presupposes that the small scale processes are both stationary and random. This may not always be the case, so *caveat emptor!*. This definition can be presented mathematically as:

$$u = \bar{u} + u' \quad (11.6)$$

where we have

$$\bar{u} = \frac{1}{T} \int_T u dt \quad (11.7)$$

which, naturally, requires that

$$\int_T u' dt = 0 \quad (11.8)$$

(You can prove this by taking the time integral of the first of these three equations.)

This conceptual separation of velocity components is referred to as “Reynolds Decomposition”. Now the same thing could be argued about the concentration of *stuff*, *i.e.*, there is some smoothly varying, mean concentration distribution coupled with some randomly varying component, so we do the same thing for C with:

$$\begin{aligned} C &= \bar{C} + C' \\ \bar{C} &= \frac{1}{T} \int_T C dt \end{aligned} \quad (11.9)$$

$$\int_T C' dt = 0 \quad (11.10)$$

in a similar fashion. Now if we plug these back into our original equation we get the following

$$\frac{\partial}{\partial t}(\bar{C} + C') = -\frac{\partial}{\partial x} [(\bar{u} + u')(\bar{C} + C')] = \frac{\partial}{\partial x} [\bar{u}\bar{C} + u'\bar{C} + \bar{u}C' + u'C'] \quad (11.11)$$

but the alert and clever student will notice that we’ve forgotten to put the molecular diffusive and J terms on the equation, but we’re being a little lazy, and will put them back in a moment (it doesn’t change anything here). Now, average the above equation with respect to time. We now have

$$\frac{\partial \bar{C}}{\partial t} + \frac{\partial}{\partial t} \left(\frac{1}{T} \int_T C' dt \right) = -\frac{\partial}{\partial x} \left[\bar{u}\bar{C} + \frac{\bar{C}}{T} \int_T u' dt + \frac{\bar{u}}{T} \int_T C' dt + \frac{1}{T} \int_T u'C' dt \right] \quad (11.12)$$

Now in doing this, we’ve taken advantage of the fact that you can reverse the order of integration and differentiation in continuous systems, and that the mean distributions are time invariant. Note also that the integrals of the fluctuating components are by definition zero, *except the last integral*. This is because there is likely to be a non-zero correlation between velocity and concentration fluctuations, since the former likely causes the latter (think about it!). In fact, it is the cross-correlation function between velocity and concentration and will in general, not be zero. We can therefore simplify the above equation to be:

$$\frac{\partial \bar{C}}{\partial t} = -\frac{\partial}{\partial x}(\bar{u}\bar{C}) - \frac{\partial}{\partial x}(\overline{u'C'}) \quad (11.13)$$

where the first term on the right hand side is the “macroscopic” advective flux divergence, and the second term is the divergence of the **Reynolds Flux**. The “overbar” refers to time averaging, as we have defined it for the velocity and concentration earlier. As a side note, if the *stuff* we were dealing with was momentum, then the equivalent term would be “Reynolds Stress”.

Now where can we go from here? Well, as we mentioned, the cross-correlation must be causal in nature, since random displacements in the fluid result in apparent concentration anomalies. We would argue that in a randomly moving fluid, there is a characteristic space scale of displacement, which we'll call ℓ' . This may be the mean vertical motion of a fluid parcel caused by breaking internal waves, or the horizontal movement caused by eddies sweeping by. Now this random displacement, coupled with a large scale mean gradient in concentration will result in an apparent concentration fluctuation C' governed by:

$$C' = -\ell' \frac{\partial \bar{C}}{\partial x} \tag{11.14}$$

Note the sign: if the slope is negative, a positive displacement results in a positive concentration anomaly, and if the slope is positive, a positive displacement results in a negative concentration anomaly. Figure 11.1 shows this schematically. We suggest you look at a paper discussing the mixing length concept by Chris Garrett (1989). Thus we have:

$$\overline{u'C'} = -\overline{u'\ell'} \frac{\partial \bar{C}}{\partial x} \tag{11.15}$$

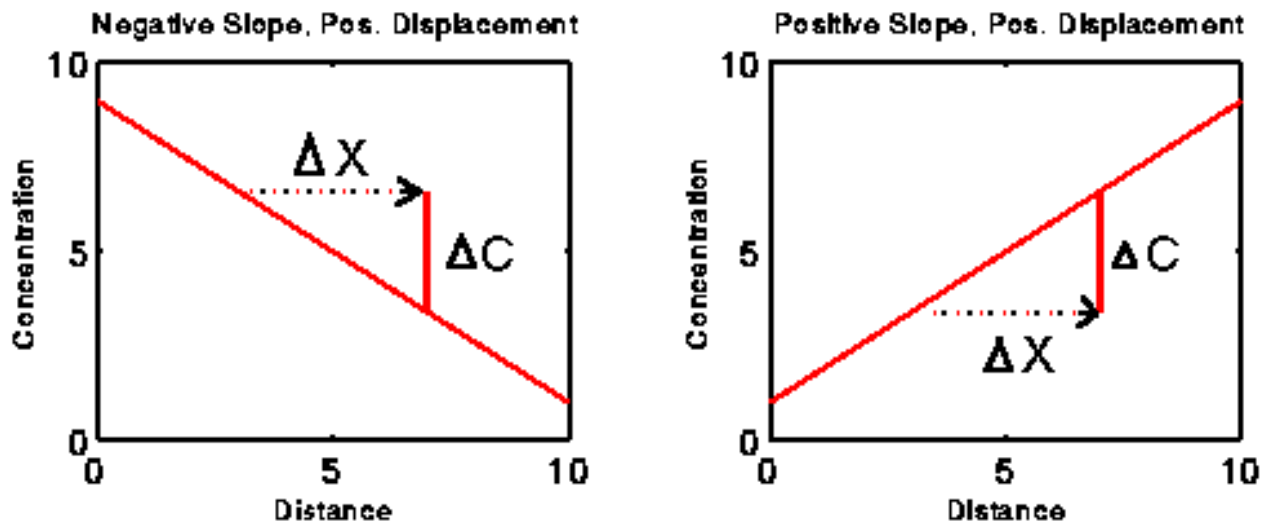


Figure 11.1: A schematic display of the effect of turbulent displacement.

Now this is another important step, since we are now separating the causes of random concentration fluctuations into two components: one due to the large scale distribution of C (thus a concentration dependent part) and one due to the fluid motion (and hence not related to C).

Thus we now have the following equation:

$$\frac{\partial \bar{C}}{\partial t} = -\frac{\partial}{\partial x}(\bar{u}\bar{C}) + \frac{\partial}{\partial x} \left[(\overline{u'\ell'} + D) \frac{\partial \bar{C}}{\partial x} \right] + J \quad (11.16)$$

where we have now come clean and put back in the molecular diffusion and J terms. Whew! Notice that we have sneakily put the $u'\ell'$ term in with the molecular diffusion term since they are functionally similar in form.

Now what is this $u'\ell'$ thing? It is a property of the fluid flow (not of the fluid, and not of the *stuff* that we studying). It is often called ***the turbulent diffusivity coefficient*** since it appears in the equation like a diffusivity term.

We mentioned that it is a property of the fluid flow. It's related to the cross-correlation between fluid displacement and the velocity. One would expect them to be more-or-less correlated in a turbulent fluid (therefore the cross-correlation will not be zero!), but not perfectly, since that only occurs for wave motions. One might express this quantity as the product of the rms (root mean square) displacement times the rms velocity fluctuation, times the cross-correlation. Typically, the cross correlation is around $\frac{1}{4}$ or so for most turbulent fluids.

So how big is the size of this $u'\ell'$ term? In the thermocline, vertical motions associated with internal waves and tides of order of meters to 10's of meters are not uncommon. This displacement occurs on time scales of hours, so associated velocities are of order 10^{-4} m/s. The product of these two yield of order 10^{-5} m²/s, which is many orders of magnitude larger than molecular diffusion. In fact, measurement of this property, by various means, yields values of order 10^{-5} m²/s in the main thermocline, and larger in the mixed layer and the abyss.

How about horizontal motions? The disparity becomes even larger. In the open ocean, velocity fluctuations of order 1-10 cm/s are common, and the space scale of eddies is of order 10-100 km. Thus horizontal eddy diffusion coefficients are of order 100-1000 m²/s. So we may as well throw D away. Well, maybe not always, since there are cases (particularly in lakes) where vertical turbulent diffusion may be suppressed by density stratification to a level where molecular diffusion may dominate. We'll discuss this in a later section. For the moment, however, we will ignore D and rewrite the equation as:

$$\frac{\partial C}{\partial t} = -\frac{\partial}{\partial x}(uC) + \frac{\partial}{\partial x} \left(\kappa \frac{\partial C}{\partial x} \right) + J \quad (11.17)$$

where we have dropped the overbars (the averaging symbols) under the assumption that we are dealing with the large scale mean velocities and concentrations. Also we have introduced a turbulent diffusivity coefficient which *is a property of the fluid flow, not of the fluid itself so that it may possibly change in space and time.*

11.4 The Numbers Game

Before going on to talk about specific oceanographic results, we will digress to discuss various NUMBERS that are of interest in oceanography. Numbers are, by definition, dimensionless quan-

ties which somehow embody physically important characteristics of the systems being studied. More often than not, they indicate the relative importance of various processes, and appear as the ratio of either terms or time scales. The reason for describing your system in terms of these numbers is that often the character of model output tends to depend more on the ratio of terms rather than their absolute value. So, for example, when faced with modeling the behavior of a system over a wide range of velocities and diffusivities, we may find that we need only perform experiments over a range of the *ratio* of two terms (the *Peclet number*) rather than doing every conceivable velocity and diffusivity combination. We will list some of the more important, along with typical values and what they mean.

11.4.1 The Reynolds Number

The Reynolds number, often referred to as Re is a measure of the relative importance of inertial to viscous terms. The higher the Reynolds number, the more likely to be turbulent the flow, and the lower the number, the more likely the flow will be laminar. The Reynolds number is defined as:

$$Re \equiv \frac{uL}{\nu} \quad (11.18)$$

where the denominator is the kinematic viscosity, denoted by the greek letter “nu” (ν). This may also be regarded as the ratio of the (molecular) diffusive time scale to the advective time scale. For most fluids (the atmosphere, oceans, *etc.*) the Reynolds number is of the order of several thousand or greater. Thus most fluids are in a state of turbulent flow.

11.4.2 The Peclet Number

The Peclet number (Pe) is a measure of the relative importance of advection to diffusion. Diffusion here is turbulent diffusion. The higher the Peclet number, the more important is advection. It is given by:

$$Pe \equiv \frac{uL}{\kappa} \quad (11.19)$$

and can be arrived at by non-dimensionalizing the advection-diffusion equation. This number may also be thought of as the ratio between the diffusive to the advective time scales. A typical open ocean is characterized by velocities of order .01 m/s, lengths of order 2-3000 km (the size of ocean gyres), and turbulent diffusivities of order 1000 m²/s. This gives a Peclet number of order 20-30.

The trick, though, is in the seemingly arbitrary choice of the length scale L . Clearly, the bigger L becomes, the higher the Peclet number becomes. This is equivalent to saying that given enough time, advection *always* wins out over diffusion. This is because while the displacement of a particle increases linearly with time with advection, it only increases as the square root of time with diffusion. This can be seen by thinking about diffusion as a random walk experiment (which is what it mathematically is). But it does boil down to this implicit ambiguity that the Peclet number

(and hence the apparent relative role of advection and diffusion) depends on the spatial scale of the system being studied.

Radioactive tracers, with their built in decay constants can define their own space scales. This can also be seen by non-dimensionalizing the advective-diffusive-decay equations. The characteristic length scale is the velocity divided by the decay constant, or quite simply the distance a fluid parcel would go before the tracer would be reduced to $1/e$ of its value by decay. Thus the *radiotracer Peclet number* would be defined as:

$$Pe = \frac{u^2}{\kappa\lambda} \quad (11.20)$$

Now for a given fluid flow, the length scale will be different for differing radiotracers, so that diffusion and mixing will be more important for one tracer than for another. Consider, for example ^7Be , which has a half life of 53.4 days, and thus has a decay probability of $1.51 \times 10^{-7} \text{ s}^{-1}$. For the subtropical North Atlantic, with velocities of order .01 m/s, and horizontal turbulent diffusivities of order $1000 \text{ m}^2/\text{s}$, this gives a Peclet number of order 0.7, which says that diffusion and mixing are as/more important than advection. Consider the same situation, however, with tritium (half-life 12.45 years). The same calculation yields a Peclet number of order 50-60, which says that tritium is more affected by advection than diffusion. Now let's turn the problem around and say that if you were interested in studying the effects of diffusion, you'd be more interested in using ^7Be than tritium.

11.4.3 The Richardson Numbers

The ocean is in general stably stratified. That is, heavy water is overlain by lighter water. If the reverse were true, then the water column would be gravitationally unstable, and vertical motions (convection) would result that would erase the condition. Now for turbulent displacement to occur vertically in a stratified water column, the fluid particles must overcome the vertical density (buoyancy) gradient. (We'll discuss this more in the next section). Thus one would expect that the ability of the water column to resist this vertical turbulence will be related to the vertical density gradient (also referred to as the rate of buoyancy production). Now one model of the origin of the energy required to produce turbulent motions is the *vertical shear in the horizontal velocity*. That is, if the horizontal velocity is changing with depth, the different layers traveling at different speeds tend to "rub" against one another, and there must be an overall dissipation occurring to maintain the velocity gradient. This dissipation scales as the square of the velocity gradient, so that defines a *Richardson Flux number* as:

$$R_f \equiv \frac{\gamma g \frac{\partial \rho}{\partial z}}{\nu \rho_0 \left(\frac{\partial u}{\partial z} \right)^2} \quad (11.21)$$

where γ is the thermal conductivity, ν is the kinematic viscosity, g is the gravitational constant and u is the horizontal velocity. This is the ratio of buoyancy production to turbulent kinetic energy. Another important quantity is the **gradient Richardson number**, which is defined by:

$$R_g \equiv \frac{g \frac{\partial \rho}{\partial z}}{\rho_o \left(\frac{\partial u}{\partial z} \right)^2} \quad (11.22)$$

In situations where R_g decreases much below 1, turbulent diffusion becomes important, and can grow to a point where the system mixes vigorously. Laboratory experiments indicate that a critical R_g of 0.25 is a good approximation for most systems.

11.4.4 Various Other Numbers

Various other numbers crop up in different circumstances. Ones that you may hear of are the **Prandtl number** and the **Schmidt number**. The former is the ratio of viscosity to thermal diffusion:

$$Pr \equiv \frac{\nu}{\gamma} \quad (11.23)$$

and the latter is the ratio of viscosity to molecular diffusion

$$Sc \equiv \frac{\nu}{D} \quad (11.24)$$

These are often used to compare model or flux calculations between different situations or chemical species.

11.5 Vertical Turbulent Diffusion

The gradient Richardson Number tells us something about the vigor of vertical turbulent diffusion in the environment. Typical estimates of vertical turbulent diffusion tend to vary inversely as the vertical density gradient. For example, consider the so called “mixed layer” of the ocean, which is generally isothermal (and hence uniform in density) and stirred effectively by wind stress. Vertical mixing rates are of order 10^{-3} to 10^{-2} m²/s. In fact, many upper ocean models tend to regard the mixed layer as so well mixed that they treat it as *perfectly mixed*: such models are called “bulk mixed layer models”.

In the main oceanic thermocline, which is characterized by vertical density gradients of order 10^{-3} kg/m⁴ (that is, a change in density of about 1 kg/m³ over a depth range of approximately 1 km) turbulent diffusion is much less vigorous, and is typically thought of to be of order 10^{-5} m²/s.

11.5.1 The Brunt-Väisälä Frequency

Most wisdom tends to view vertical turbulent diffusivity as being an inverse function of the *Brunt-Väisälä* (spelling optional) frequency N . This is a measure (in inverse time units) of the stratification or resistance to turbulence. The derivation of this concept is actually quite simple. Imagine displacing a fluid particle vertically in a stably stratified water column. If the particle were to be lifted a distance Δz , it would be denser than the surrounding water and experience a downward force related to the difference in density between it and the surrounding water:

$$F = g\Delta\rho = -g\Delta z \frac{\partial\rho}{\partial z} \quad (11.25)$$

which in turn would be a function of the distance of displacement and the density gradient. The equation applies for negative displacements as well, since the particle would be more buoyant than the surrounding water, and want to “cork” back up. Now this restoring force which is proportional to distance should look familiar, to those of you who have taken high school physics: it is the spring equation. If we used the fact that $F = ma$ (a is the acceleration), then we can rewrite the equation (using the density ρ in place of the mass) to have:

$$\rho a = -gz \frac{\partial\rho}{\partial z} \quad (11.26)$$

or

$$\frac{\partial^2 z}{\partial t^2} = -\frac{g}{\rho} \left(\frac{\partial\rho}{\partial z} \right) z \quad (11.27)$$

where the terms in front of the z on the right hand side of the equation can be expressed as a constant, which we’ll call N^2 . The solution to this equation is simply:

$$z(t) = z_0 e^{iNt} \quad (11.28)$$

a simple harmonic oscillator with frequency N ,

$$N = \sqrt{\frac{g}{\rho} \frac{\partial\rho}{\partial z}} \quad (11.29)$$

the *Brunt-Väisälä* Frequency. That is, in the absence of dissipation, if you were to “pluck” a fluid particle above its equilibrium position, it would return to its position, but overshoot, oscillating back and forth with a frequency N .

This frequency is a measure of the water column stability, and is typically a few cycles per hour (10^{-3} Hz) in the main thermocline. Surveys by Sarmiento *et al.* (1976), Gargett (1984), and Gregg (1987) attempt to correlate apparent diffusivities with N and other parameters. The problem is extraordinarily complex, and not well resolved, but the following may be regarded as the typical ranges of diffusivities:

- ocean mixed layer, unstratified and strongly mixed: 10^{-3} to 10^{-1} m²/s

- main oceanic thermocline, stratified, weak mixing: 10^{-5} m²/s deep ocean
- away from boundaries, weak stratification: 10^{-4} m²/s near bottom,
- especially near boundaries: 10^{-4} to 10^{-3} m²/s chemically, salinity or
- thermally stratified lakes or estuaries: 10^{-9} to 10^{-6} m²/s

11.6 Horizontal Turbulent Diffusion

Horizontal motions are less inhibited by stratification. This is evident when you think of the aspect ratio of the oceans. Density contrasts which occur over a few hundred meters in depth are reflected in changes which occur over thousands of kilometers. The typical scales of motion associated with horizontal turbulence are the “eddy scale” which is of order 10-100 km. The kinds of estimates that you see for eddy diffusion range from a few m²/s to a few thousand m²/s, depending on location and scale.

The scale is important, because the kind of averaging implicit in the parameterization of the eddy diffusion, *i.e.* what you call \bar{u} and what you call u' depends on the scales over which you are diffusing your material. This can be seen in a classic paper (we recommend you read it, however old it is) by Okubo (1971). For a two dimensional gaussian and a constant turbulent diffusivity, you would expect the mean width to scale with the square root of time, or that the variance (the square of the width) to vary linearly with time. Comparison with observed experiments, however, is not so simple, because the distributions quickly become distorted by motions and shearing on scales comparable to the size of the patch (remember, there is not really any actual scale separation, there is always a range of motions between the u' and the \bar{u} ranges). The best you can do is to define some r.m.s. (root mean square) width as:

$$\sigma_{rC}^2 \equiv \frac{\int_0^{\infty} r^2 C(t, r) dr}{\int_0^{\infty} C(t, r) dr} \quad (11.30)$$

i.e., the second moment of the distribution (which obviously will not be symmetric or gaussian). The argument is that in a kind of “central limit” sense, this will represent the mean spreading of the patch, and that it should behave as a gaussian in some mean sense. However, because the increasingly larger (with time) dye patch feels larger and larger motions as “turbulent”, the diffusivity should increase with time.

What is actually observed is a much stronger than linear dependence of the mean square radius with time, as seen in the plot in Fig. 11.2 (from Okubo’s (1971) paper).

Think of the dye patch spreading in the open ocean. Clearly, when the dye patch is small, only those turbulent motions small compared to the dye patch itself can play a role in its dispersal. However, as the patch grows, the range of turbulent motion accessible to the patch for dispersal grows, so that the amount of available energy put into mixing grows with time and size. Thus there

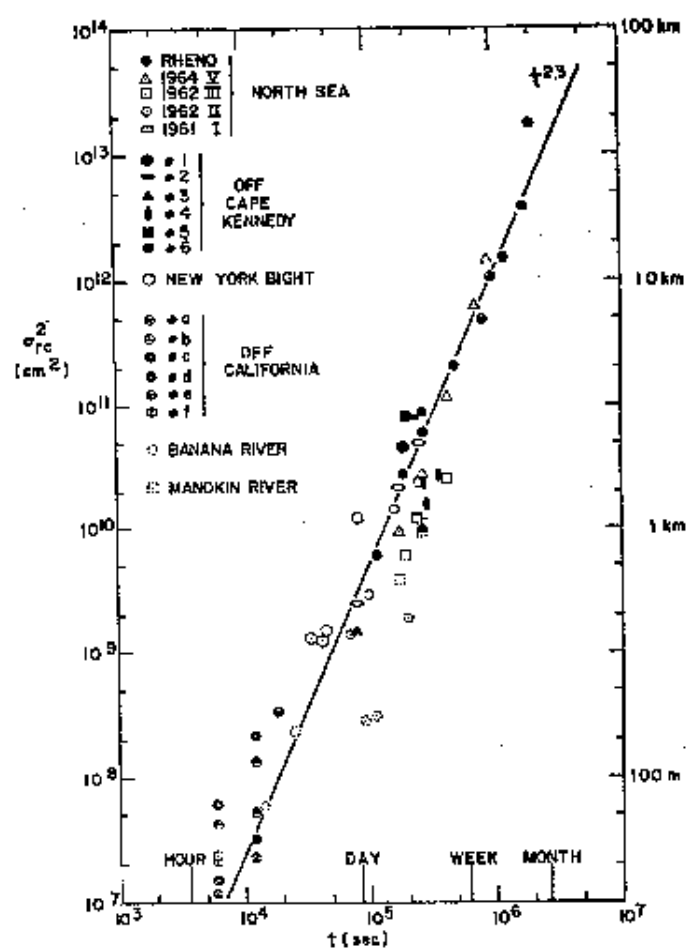


Figure 11.2: The rms of concentration as a function of time (from Okubo, 1971).

is an apparent increase in horizontal turbulent diffusivity with scale. Using the observed spreading, Okubo estimates an approximately $\ell^{1.1}$ dependence of the diffusivity on length scale, shown here in Fig 11.3

The take-home lesson is obvious. If you are looking at the dispersion of a small dye patch in the ocean, then it will exhibit small diffusivity initially (perhaps 0.1 to 1 m^2/s) but as it grows in size, diffusivity grows. On ocean basin scales, you would expect diffusivities of order 100 - 1000 m^2/s . Diffusivity is in the eye of the beholder.

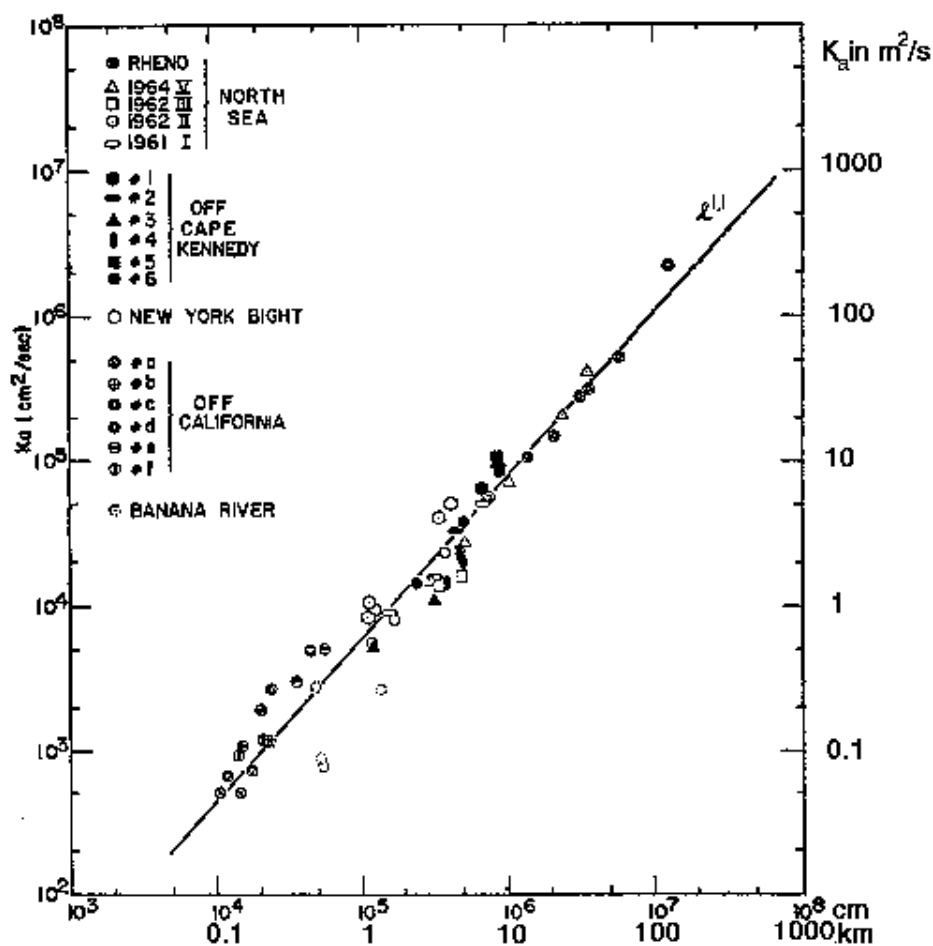


Figure 11.3: Horizontal eddy diffusivity as a function of horizontal scale (from Okubo, 1971).

11.7 The Effects of Varying Turbulent Diffusivity

We rewrite the equation in the following way:

$$\frac{\partial C}{\partial t} = -u \frac{\partial C}{\partial x} + \frac{\partial \kappa}{\partial x} \frac{\partial C}{\partial x} + \kappa \frac{\partial^2 C}{\partial x^2} + J \quad (11.31)$$

where we have differentiated by parts the turbulent diffusion term. We have admitted the possibility that the diffusivity may vary in space because it is a property of the flow, not of the fluid.

The second term on the R.H.S. looks a lot like a velocity. In fact, it behaves exactly like one. How important is it? Well, consider first horizontal effects, since those diffusivities are largest. Values as high as $2000 \text{ m}^2/\text{s}$ have been observed in the Gulf Stream area, and seem to taper off to

a few hundred m^2/s in the ocean interior. Assuming this takes place over a distance of 2000 km yields apparent “velocities” of approximately $(2000 \text{ m}^2/\text{s})/(2 \times 10^6 \text{ m}) = .001 \text{ m/s}$, or about an order of magnitude or more smaller than typical velocities in the upper thermocline. So the conclusion is that this is unlikely a problem for the upper thermocline, but may be important deeper down where velocities are small.

Now how about the vertical? Well in the mixed layer, vertical diffusivities are of order 10^{-3} or larger. We’ll choose the lowest values for this calculation. Meanwhile, a few hundred meters down in the main thermocline, vertical diffusivities are of order $10^{-5} \text{ m}^2/\text{s}$, so we calculate an effective “velocity” of order 10^{-6} to 10^{-5} m/s . This doesn’t sound like much, but when you realize that typical Ekman pumping rates (the rate at which water is pushed downward by wind stress convergence) are of order 50 m/y or about 10^{-6} m/s , this becomes really significant! Furthermore, abyssal upwelling velocities through the main thermocline (we’ll get into that in chapter 13) are ten times lower. Thus we should be very careful when considering vertical balance models with turbulent diffusivities that change a lot with depth.

Now a footnote: the above considerations may be a little premature, since the physics associated with models that span large ranges in turbulent flows (which must drive these changes in turbulent diffusivities) is far from worked out. Our goal in pointing out these issues to you at this point is to make you aware of some of the pitfalls and foibles in such modeling.

A few papers which discuss variable diffusivity effects are: Armi, L. (1979) horizontal; Armi, L. and D. Haidvogel (1982) horizontal; Jenkins, W.J. (1980) vertical.

11.8 Problems

All of your problems sets are served from the web page:

http://eos.whoi.edu/12.747/problem_sets.html

which can be reach *via* a number of links from the main course web page. In addition, the date the problem set comes out, the date it is due, and the date the answers will be posted are also available in a number of locations (including the one above) on the course web page.

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